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Identification problems in the solution of linearized DSGE models*

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Abstract

This article analyzes identification problems that may arise while linearizing and solving DSGE models. A criterion is proposed to determine whether or not a set of parameters is partially identifiable, in the sense of Canova and Sala (2009), based on the computation of a basis for the null space of the Jacobian matrix of the function mapping the parameters with the coefficients in the solution of the model.

Keywords: Parameter identification, DSGE models
JEL Classification: C13, C51, C52, E32

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1 Introduction

As Canova and Sala (2009) have pointed out, not enough attention has been devoted to analyze identifiability in the context of DSGE models estimation. This is surprising since many researchers interested in matching these models with the data have recently turned their attention to likelihood based estimation methods, mainly, to Bayesian estimation techniques. A notable exception to this peculiar omission is the article by Canova and Sala (2009) in which they highlight and classify many of the identification issues that may arise while estimating DSGE models and propose various tools to detect them.

Based on an objective function measuring the distance between the model impulse - response functions and those obtained from a structural VAR, Canova and Sala (2009) used a standard RBC model as a preliminary example to illustrate some common features related to parameter identification. To check for partial identification, they fix most of the parameters of the model and plot the surface and the contour sets of the objective function, varying only two parameters at a time. Such analysis of the objective function shows that it is flat or nearly flat in some subsets of its domain (again, considered as a function of two parameters only). This approach to recognize partial identification is clearly useful when a data set is considered and a real valued objective function has been previously defined, but a more general method can be proposed for the case where such problems arise directly from the solution of the model. In fact, Canova and Sala go further and observe that some parameters are partially unidentifiable because their individual effects on the coefficients of the matrices expressing the solution of the linear model are proportional. To do so, they compute the partial derivatives of the function mapping the parameters to the coefficients of the solution of their RBC model.

Pointing in the same direction, Iskrev (2007) applies the decomposition of the information matrix, proposed by Rothenberg (1966), to separate those identification issues that come from the structure of the model from those that appear once some data is considered. As Canova and Sala mention, usually unidentifiability is caused by an ill-behaved mapping between the parameters and the coefficients of the solution of the model; hence they propose to compute the rank of the Jacobian matrix of that mapping as a criterion to determine if some parameters cannot be identified, no matter what objective function is used for the estimation or how much information is considered.

This article focuses exclusively on identification issues that arise in the process of solving the model. It is a common practice to solve DSGE models by (log)linearizing them around some specific point, usually their steady state, and then applying a numerical algorithm to solve linear rational expectations models. The coefficients in the solution of the model end up being non linear functions of its parameters; some of this functions could be non injective giving rise to unidentifiable parameters. A direct consequence of the inverse function theorem is that the function mapping the parameters with the solution of the model would be locally injective if its Jacobian matrix had full column rank (assuming that the number of coefficients in the matrices expressing the solution is greater than the number of parameters).

Iskrev (2008) proposes a method to obtain this Jacobian analytically for the case of a linearized DSGE model whose solution has been computed numerically. Since the computation of the rank of a matrix is highly sensitive to errors due to numerical precision, if possible, the use of analytical derivatives is recommended. Nonetheless, if the steady state of the model cannot be computed analytically, Iskrev's method cannot be fully applied either[1]. Unfortunately, many DSGE models intended to be empirically evaluated, such as those used world wide at the Central Banks for policy analysis and forecasting, are complex enough to prevent the analytic calculation of their steady state. But even in those cases, Iskrev's application of the implicit function theorem can be complemented with the chain rule for multivariate functions, as shown in the second section of this article, to considerably reduce the amount of numerical computations in the derivation of the Jacobian.

In section 3, the problem of partial identification due to the solution of the model is considered and a criterion for detecting it is proposed, based on the calculation of a basis for the null space of the Jacobian matrix of the function mapping the parameters with the solution of the model. The main idea behind the test is that some vector in the basis of this null space has a non-zero entry in its i-th coordinate if and only if the corresponding column of the Jacobian matrix has a non-zero coefficient in a zero linear combination of those columns. Intuitively, this criterion points out those parameters in the model that are responsible for a rank deficient Jacobian and thus it can be used to find partially identifiable parameters in the sense of Canova and Sala (2009). More specifically, in virtue of Proposition 2, the criterion proposed allows us to easily build maximal sets of identifiable parameters, i.e., parameters whose identification problems, if any, do not come from the structure of the model.

2 The solution of the model

Following Uhlig (1995), a (log)linearized DSGE model can be written as:

$$\begin{aligned} E_t \left[F x_{t+1} + G x_t + J x_{t-1} + L z_{t+1} + M z_t \right] &= 0 \\ z_{t+1} = N z_t + \epsilon_{t+1}; \quad E_t \left[\epsilon_{t+1} \right] = 0 & \end{aligned} \tag{1}$$

where x_{t-1} is a $m \times 1$ vector of endogenous state variables, z_t is a vector of exogenous stochastic processes of size $n \times 1$ and ϵ_t can be assumed independent, identically and normally distributed. The coefficient matrices F, G, J, L, M and N are nonlinear functions of a $k \times 1$ vector, θ, containing the parameters of the nonlinear DSGE model. Iskrev (2008) introduces a convenient notation that we will follow here. To begin with, the $s \times 1$ vector containing the coefficients of the matrices in the structural model (1) is denoted by γ, so clearly γ is a nonlinear function of the deep parameters θ.

[1]This observation was made to me by Andrés González while jointly estimating a DSGE model for the Colombian economy proposed by González et al. (2009)

According to Uhlig (1995), the solution to (1) can be written as a recursive equilibrium law of motion :

$$x_t = Px_{t-1} + Qz_t \tag{2}$$

where matrices P and Q are assumed to be such that the corresponding equilibrium is stable. As in Iskrev (2008), let $\phi = \left[vec\,(P)', vec\,(Q)'\right]'$ be the $\left(m^2 + mn \times 1\right)$ vector whose components are the corresponding coefficients in (2), then the solution of the nonlinear DSGE model can be seen as a function h mapping θ to ϕ.

Equation $z_{t+1} = Nz_t + \epsilon_{t+1}$ can be lagged one period and substituted into (2) to obtain $x_t = Px_{t-1} + Q\,(Nz_{t-1} + \epsilon_t)$. Furthermore, a measurement equation can be added to the system so that a possible state space representation of the model is:

$$\begin{bmatrix} x_t \\ z_t \end{bmatrix} = \begin{bmatrix} P & QN \\ 0 & N \end{bmatrix} \begin{bmatrix} x_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} Q \\ I_{(n\times n)} \end{bmatrix} \epsilon_t \tag{3}$$

$$y_t = C \begin{bmatrix} x_t \\ z_t \end{bmatrix} + u_t$$

If ϵ_t and u_t are normally distributed, the Kalman filter can be used to obtain the conditional log-likelihood function, $l\,(y, \theta)$. Maximum likelihood and Bayesian estimation methods require the maximization of either the likelihood function or a posterior distribution.

A traditional approach to recognize unidentified parameters in a likelihood-based estimation framework is to calculate the rank of the information matrix, defined as:

$$\Im_\theta = -E\left[l_{\theta\theta}\,(y, \theta)\right]$$

There is a considerable number of methods available to compute the information matrix. For the case of a linear model with non linear restrictions, as (3), Iskrev (2008) quotes Rothenberg (1966) to remark that $\Im_\theta$ can be decomposed into the product of the information matrix of the unrestricted model and a matrix H with the derivatives of the nonlinear restrictions on the parameters.

To do so, let $\hat{P} = \begin{bmatrix} P & QN \\ 0 & N \end{bmatrix}$, and $\hat{\phi} = \left[vec\left(\hat{P}\right)', vec\left(\hat{Q}\right)'\right]'$. As Iskrev (2008) points out, θ affects $l\,(y, \theta)$ through $\hat{\phi}$ only, via a function $\hat{\phi} = \hat{h}\,(\theta)$. Furthermore, if $H = \hat{h}_\theta$ is the Jacobian matrix of $\hat{h}$ then, as shown by Rothenberg (1966), $\Im_\theta$ can be calculated by:

$$\Im_\theta = H'\Im_{\hat{\phi}}H \tag{4}$$

where $\Im_{\hat{\phi}}$ is the information matrix of the unrestricted reduced-form model (3).

Iskrev (2008) mentions several authors who have proposed analytical methods to derive the information matrix of a linear state space model, so it is not necessary to name them again here. His contribution is to propose an analytical expression for H, when it can be seen as the Jacobian of a function expressing the solution of a nonlinear DSGE model.

Usually, the solution of such a model cannot be computed analytically. In fact, many recent papers in the field are concerned with the problem of finding an efficient algorithm to derive it numerically, as those proposed by Blanchard and Kahn (1980), Anderson and Moore (1985), Uhlig (1995), Klein (2000) and Sims (2002) among others. This means that in many of the cases of interest, we can only obtain a numerical representation of the solution of the model (2) and, of course, of its state space form (3). Insofar as the functions h and $\hat{h}$ cannot be obtained analytically, an analytical expression for H cannot be obtained by direct non-numerical differentiation of these functions. Hence, Iskrev (2008) proposes to apply the implicit function theorem to find H.

In fact, as stated by Uhlig (1995), since the recursive law of motion (2) solves equations (1) then the following hold:

$$(FP + G)\,P + J = 0 \tag{5}$$

$$(FQ + L)\,N + (FP + G)\,Q + M = 0 \tag{6}$$

Assuming that (1) has a unique solution, as Iskrev (2008) does, equations (5) and (6) can be thought as defining an implicit function $F : \mathbb{R}^{k+m^2+mn} \rightarrow \mathbb{R}^{m^2+mn}$ such that $F(\theta,\phi) = 0$ if and only if $\phi = h(\theta)$. So the implicit function theorem implies that:

$$H = -F_{\phi}(\theta,\phi)^{-1}\,F_{\theta}(\theta,\phi)$$

where $F_{\phi}(\theta,\phi)$ and $F_{\theta}(\theta,\phi)$are the corresponding Jacobian matrices of F.

It is convenient to remember, however, that the coefficients of the structural model (1), contained in γ, are functions of the vector of parameters θ obtained by (log)linearizing the nonlinear DSGE model around some point, usually its steady state. For the common case of the steady state, and for many other cases that might be interesting to consider, such point varies with θ. Therefore, γ depends on θ directly but also through the steady state, and then it can be thought as a function of the form $\gamma(\theta, ss(\theta))$. The problem here is that in most of the non trivial cases the steady state values cannot be obtained as an explicit function of the parameters θ. In fact, it is common practice to use numerical algorithms to find the steady state of a DSGE model for a given vector θ. For example, the latest versions of Dynare, a program for solving and estimating rational expectations models described in Juillard (2001), includes four different algorithms among which the user may choose in order to achieve this task. In such cases, Iskrev's method cannot be applied since some derivatives of $F(\theta,\phi)$ cannot be computed analytically, specifically those expressed in terms of the steady state values of the variables in the model.

Fortunately, the problem mentioned above can be solved partially by means of a simple application of the chain rule. Iskrev's main concern in finding an analytical expression for H is to reduce the degree of numerical error in the computation of the information matrix, as stated in Iskrev (2007), and even for the case where the steady state cannot be solved explicitly as a function of θ, most of the derivatives involved in the computation of $F_\theta(\theta, \phi)$ can still be obtained analytically. In fact, since F depends on θ only through γ, then:

$$F_\theta(\theta, \phi) = F_\gamma(\theta, \phi)\left(\gamma_\theta(\theta, ss(\theta)) + \gamma_{ss}(\theta, ss(\theta))\, ss_\theta(\theta)\right)$$

and all these derivatives can be computed analytically, except for $ss_\theta(\theta)$.

3 Partial identifiability

The fact that the information matrix can be decomposed as in equation (4) allow us to isolate those parameters that cannot be identified only due to of the specification, linearization or solution of the model, independently of the estimation criteria or the data, as Iskrev (2007) points out. In terms of Canova and Sala (2009), if matrix H does not have full column rank then we may have what they call a problem of lack of or partial identification. In particular, lack of identification issues, i.e., the existence of parameters that somehow disappear either in the linearization or in the solution of the model, are easy to recognize after computing H. In fact, since, $H_{ij} = \frac{\partial \phi_i}{\partial \theta_j}$, if a single parameter θ_j cannot be identified, the corresponding column of H is the vector 0.

Now, suppose that Θ_i is the economically relevant subset of $\mathbb{R}$ determining the values that θ_i can take, and consider the function $\tilde{h} : \Theta_{i_1} \times ... \times \Theta_{i_p} \to \mathbb{R}^{m^2+mn}$ obtained by restricting h to $\Theta_{i_1} \times ... \times \Theta_{i_p}$ and fixing all other parameters in vector θ. If $\tilde{h}$ is not injective then there are different combinations of these parameters for which the solution of the model (the coefficients in ϕ) are the same. In that case, we would have partial identification in the sense of Canova and Sala and, correspondingly, we will call $\{\theta_{i_1}, ..., \theta_{i_p}\}$ a *set of partially identifiable parameters* (SPIP). Hence, the rank of H would be less than the number of columns, k. This follows immediately from the fact that, in general, if h is a continuously differentiable function from an open subset of $\mathbb{R}^k$ to $\mathbb{R}^l$, with $k \leq l$, and its Jacobian matrix H has full column rank then it is locally injective. It should be emphasized, however, that having a full column rank Jacobian is just a sufficient but not a necessary condition for the function h to be locally injective[2], so we cannot conclude that there are unidentifiable parameters from the fact that H has not full column rank. Nevertheless, the rank of H still gives us some useful information about the identifiability of the parameters of the model. In fact, the two propositions stated and proved below provide an easy way to find maximal sets of parameters that are not partially identifiable, in the previous sense.

Note that if $\{\theta_{i_1}, ..., \theta_{i_p}\}$ is a SPIP, as defined above, then $\theta_{i_1}, ..., \theta_{i_p}, \theta_{i_{p+1}}$ are also partially identifiable. On the contrary, if one parameter is taken out from the set the case might be that the resulting set is not a SPIP, i.e., that the function h restricted to $\Theta_{i_1} \times ... \times \Theta_{i_{j-1}} \times \Theta_{i_{j+1}} ... \times \Theta_{i_p}$is locally injective. The

[2]The function $f(x, y) = (x^3, y^3)$, for example, has a singular Jacobian at $(0, 0)$; nevertheless, it is injective in all its domain

rank deficiency of H only tells us that there may be SPIPs but what would be more useful to the researcher is to know which are those sets. Furthermore, if there is such a set then any other set containing it is also a SPIP, in particular the set with all the parameters of the model; thus the search for partially identifiable parameters may be restricted to those sets such that none of its proper subsets are partially identifiable. If a SPIP has this property, we will call it a *minimal set of partially identifiable parameters* (MSPIP).

The fact that a parameter θ_i does not belong to any MSPIP means that the structure of the model does not prevent θ_i from being identified. Any possible lack of identification related to this parameter in a likelihood-based estimation process must be attributed to the data set considered. Hence, knowing which parameters have this property is as far as we can go if we are interested in finding those identification issues that come from the structure of the model.

Suppose that $S = \left\{\theta_{i_1}, ..., \theta_{i_p}\right\}$ is a SPIP; the main difficulty in establishing if S is also a MSPIP is that it has to be determined whether or not all the restricted functions resulting from removing exactly one parameter from S are injective. A sufficient condition which is easy to verify is that the corresponding Jacobian matrices of all such functions have full column rank, but again it is not a necessary condition.

Abusing notation for the sake of briefness, let's call a set of parameters of the model linearly dependent if their corresponding columns in matrix H are, properly speaking, linearly dependent. As before, a *minimal set of linearly dependent parameters* (MSLDP) is a set of linearly dependent parameters, in the previous sense, such that none of its proper subsets are linearly dependent. Note that, since the full column rank condition is sufficient for h and for any of its restrictions, $\tilde{h}$, to be injective, then every MSLDP whose elements are partially identifiable is a also a MSPIP. Consequently, at least some of the MSPIPs can be recovered if we can recognize the MSLDPs, which is clearly easier. Proposition 1 generalizes the full column rank condition to state a criterion for recognizing those parameters of the model that belong to some MSLDP. Moreover, Proposition 2 applies such criterion to obtain the intersection of all the maximal sets of linearly independent parameters, none of which is a SPIP, and thus it makes it easier to find those sets. In order to prove the propositions, two previous lemmata shall be stated beforehand.

Lemma 1. *Let h_{θ_i} be the column of matrix H corresponding to θ_i. Suppose $\left\{\theta_{i_1}, ..., \theta_{i_p}\right\}$ is a MSLDP, then there is a vector $\alpha \in \mathbb{R}^p$ such that $\sum_{j=1}^{p} \alpha_j h_{\theta_{i_j}} = 0$ and $\alpha_j \neq 0$ for all $j = 1, ..., p$.*

Proof. If we agree that the empty set is linearly independent, then the only single-element minimal set of linearly dependent vectors is the one containing the vector zero, so the conclusion of the lemma follows trivially for $p = 1$. Now suppose that $p \geq 2$ and let $\tilde{H} = \left[h_{\theta_{i_1}} \dots h_{\theta_{i_p}}\right]$ then, by assumption, there is an $\alpha \in \mathbb{R}^p$ such that $\tilde{H}\alpha = 0$. It remains to be proved that $\alpha_j \neq 0$ for all $j = 1, ..., p$, so, on the contrary, suppose that there is a j such that $\alpha_j = 0$, then $\sum_{l \neq j} \alpha_l h_{\theta_l} = 0$ and $\alpha_k \neq 0$ for some $k \neq j$, so there are $p-1$ linearly dependent vectors in $\tilde{H}$ and thus $\left\{\theta_{i_1}, ..., \theta_{i_p}\right\}$ is not a MSLDP. □

Lemma 2. *Let $v_1, ..., v_p \in \mathbb{R}^k$, with $p \geq 2$, and suppose there is an $\alpha \in \mathbb{R}^p$ such that $\sum_{i=1}^{p} \alpha_i v_i = 0$ and $\alpha_j \neq 0$ for some j . If $v_1, ..., v_{j-1}, v_{j+1}, ..., v_p$ are also linearly dependent then there is a $\hat{j} \neq j$ and $p-1$ real numbers $\beta_1, ..., \beta_{\hat{j}-1}, \beta_{\hat{j}+1}, ..., \beta_p$ such that $\sum_{i \neq \hat{j}} \beta_i v_i = 0$ and $\beta_j \neq 0$.*

Proof. By assumption, $v_1, ..., v_{j-1}, v_{j+1}, ..., v_p$ are linearly dependent so there are $\gamma_1, ..., \gamma_{j-1}, \gamma_{j+1}, ..., \gamma_p \in \mathbb{R}$ such that $\sum_{i \neq j} \gamma_i v_i = 0$ and $\gamma_{\hat{j}} \neq 0$ for some $\hat{j} \neq j$. Therefore, $v_{\hat{j}} = \sum_{i \neq j, i \neq \hat{j}} \left(-\frac{\gamma_i}{\gamma_{\hat{j}}}\right) v_i$ and then $\sum_{i=1}^{p} \alpha_i v_i = \sum_{i \neq j, i \neq \hat{j}} \alpha_i v_i + \alpha_{\hat{j}} v_{\hat{j}} + \alpha_j v_j = \sum_{i \neq j, i \neq \hat{j}} \beta_i v_i + \alpha_j v_j$, where $\beta_i = \alpha_i - \alpha_{\hat{j}} \frac{\gamma_i}{\gamma_{\hat{j}}}$ for all $i \in \{1, ..., k\} - \left\{j, \hat{j}\right\}$. Finally, let $\beta_j = \alpha_j \neq 0$, then $\sum_{i \neq \hat{j}} \beta_i v_i = \sum_{i=1}^{p} \alpha_i v_i = 0$ and the proof is complete. □

Proposition 1. *Let B_H be a basis for the null space of the Jacobian matrix H. $b_i = 0$ for all $b \in B_H$ if and only if parameter θ_i does not belong to any MSLDP.*

Proof. Let $\mathcal{N}(H) = \left\{x \in \mathbb{R}^k : Hx = 0\right\}$ be the null space of H, and $B_H \subset \mathcal{N}(H)$ a set of linear independent vectors that spans $\mathcal{N}(H)$, i.e., a basis for $\mathcal{N}(H)$. Suppose that there is an $i \in \{1, ..., k\}$ such that $b_i = 0$ for all $b \in B_H$, then, since every $x \in \mathcal{N}(H)$ is a linear combination of the elements of B_H, $x_i = 0$ for all $x \in \mathcal{N}(H)$. On the other hand, if H does not have full column rank, i.e., if its columns are linearly dependent then, by definition, there is a vector $\alpha \in \mathbb{R}^k$, with $\alpha_j \neq 0$ for some $j \in \{1, ..., k\}$, such that $H\alpha = 0$, so $\alpha \in \mathcal{N}(H)$ and, given our assumption on B_H, $j \neq i$. In other words, there is no linear combination of the columns of H, denoted by h_{θ_l}, such that $\sum_{l=1}^{k} \alpha_l h_{\theta_l} = 0$ and $\alpha_i \neq 0$. Therefore, by Lemma 1, if $b_i = 0$ for all $b \in B_H$ there is no minimal set of linearly dependent parameters containing θ_i.

To prove the other direction, suppose that there is an $i \in \{1, ..., k\}$ such that $b_i \neq 0$ for some $b \in B_H$. Let $\tilde{b}$ be a vector of size p obtained by removing all zero components from b, and, without loosing generality, assume that we have reorganized its components so that $\tilde{b}_p = b_i$. Note that if $p = 1$ then $\theta_i = 0$ which clearly belongs to a MSLDP. If $p \geq 2$, then there are parameters $\left\{\theta_{i_1}, ..., \theta_{i_{p-1}}\right\}$ such that $\sum_{l=1}^{p-1} \tilde{b}_l h_{\theta_{i_l}} + \tilde{b}_p h_{\theta_i} = 0$. If for any proper subset of $P = \left\{\theta_{i_1}, ..., \theta_{i_{p-1}}, \theta_i\right\}$ of size $p-1$ the corresponding parameters are linearly independent, then P is a MSLDP to which θ_i belongs. So let's assume there is at least one of these proper subsets whose elements are linearly dependent. It is intuitively clear that we could always remove as many parameters as necessary from P, one at a time, until we get a MSLDP. Furthermore, as far as the size of the set considered is not less than 2, Lemma 2 guarantees that we can do so without getting rid of θ_i; therefore θ_i belongs to the resulting MSLDP. □

Proposition 2. *Suppose that the rank of H is $r \geq 1$. Parameter θ_i does not belong to any MSLDP if and only if it belongs to all the sets containing r linearly independent parameters of the model.*

Proof. Suppose that θ_i does not belong to any MSLDP but that there are r linearly independent parameters distinct from θ_i, denoted by, $\theta_{i_1}, ..., \theta_{i_r}$. Since the rank of H is r, parameters $\theta_{i_1}, ..., \theta_{i_r}, \theta_i$ must be linearly dependent, so there are $\alpha_1, ..., \alpha_{r+1} \in \mathbb{R}$ such that $\sum_{j=1}^{r} \alpha_j h_{\theta_{i_j}} + \alpha_{r+1} h_{\theta_i} = 0$ and $\alpha_j \neq 0$ for some $j \in \{1, ..., r+1\}$. However, by Proposition 1, $\alpha_{r+1} = 0$ so $j \neq r+1$. Hence, $\sum_{j=1}^{r} \alpha_j h_{\theta_{i_j}} = 0$ and $\alpha_j \neq 0$ for some $j \in \{1, ..., r\}$ which contradicts the fact that $\theta_{i_1}, ..., \theta_{i_r}$ are linearly independent.

Now, suppose that $\theta_i \in M$, where M is a MSLDP. If $h_{\theta_i} = 0$ then is obvious that θ_i does not belong to any set containing r linearly independent parameters of the model. Thus, let $h_{\theta_i} \neq 0$, it follows that the nonempty set $S = M - \{\theta_i\}$ is linearly independent. If $|S| = r$, there is a set of r linearly independent parameters to which θ_i does not belong. So let's assume that $|S| < r$, since the rank of H is r, S cannot be maximal, i.e., there is a set of linearly independent parameters properly containing S. Moreover, there must be a set R of r linearly independent parameters such that $S \subset R$. However, $M \nsubseteq R$ because M is linearly dependent, so $\theta_i \notin R$. In any case, if θ_i belongs to a MSLDP it does not belong to some set containing r linearly independent parameters. □

4 Conclusion

Insofar as the sole solution of the model is concerned, the problem of partial identifiability could be expressed in terms of the existence of minimal sets of parameters such that the function whose values are the coefficients of the solution of the model, restricted to those parameters, is non-injective. If we are interested in finding the minimal sets of linearly dependent parameters, as a previous attempt to find minimal sets of partially identifiable parameters, after computing a basis for the null space $\mathcal{N}(H)$ our search will be considerably reduced. In fact, Proposition 1 provides a criterion to determine exactly which parameters of the model belong to some MSLDP. Then, any combination of these parameters could be tested using the conventional full rank condition to determine whether or not it is a MSLDP. In the other hand, according to Proposition 2, the computation of a basis for the null space can be used to determine exactly which parameters belong to all maximal sets of linearly independent parameters and, thus, it facilitates the search for these sets. Furthermore, since the Jacobian matrix of the restricted function corresponding to any of these sets has full column rank, the restricted function is locally injective. Therefore, even if we cannot easily recognize MSPIPs, we can still use Proposition 2 to find sets of parameters which do not have the problem of being partially identifiable due to the structure of the model, moreover, to find those among such sets that are maximal in the sense that if any other parameter is added to the set, the corresponding Jacobian matrix would not have full column rank.

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